Chapter 13. Existence, Uniqueness and Stability of Stochastic nonlinear differential equations with piecewise constant arguments (SEPCA).

(SEPCA)

$$dx(t) = f(t, x(t), \lambda_{\rho(t)}(x(r(t))))\, dt$$ Ch 12

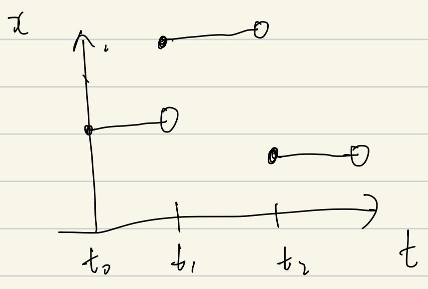
$$+ g(t, x(t), \lambda_{\rho(t)}(x(r(t))))\, dW(t)$$

$$x(t_0) = x_0.$$

$\rho(t)$, $r(t)$: piecewise constant. $K = \{k\}_{k=0}^{\infty}$, $\mathcal{T} = \{t_k\}_{k=0}^{\infty}$. in $t \in [t_k, t_{k+1})$

Ex), $t \in [t_0, t_1) \Rightarrow \rho(t) = 0$

$r(t) = t_0.$



$\lambda_0(x(t_0))$ $\lambda_2(x(t_2))$

$\lambda_1(x(t_1))$

For $t \in [t_k, t_{k+1})$, we can write

$$x(t) = x^0 + \int_{t_0}^{t} f(s, x(s), \lambda_k(x(t_k)))\, ds$$
$$+ \int_{t_0}^{t} g(s, x(s), \lambda_k(x(t_k)))\, dW(s) \qquad (13.3)$$

1. Existence. (Linear growth of f, g)
2. Comparison method
3. Stability analysis.

Def. 13.1. A stochastic process $x : (a,b) \to \mathbb{R}$ solution of (SEPCA) if

i) $x(t)$: conti, $\mathcal{F}_t$ - adapted, $\forall t \in (a,b)$

ii) $f(t, x(t), \lambda_k(x(t_k))) \in L_{ad}(\Omega, L^1(a,b))$

$g(t, x(t), \lambda_k(x(t_k))) \in L_{ad}(\Omega, L^2(a,b))$

iii) (13.3) holds, w.p. 1.

(Def 2.26 in p32)

$\mathcal{F}_t$ - adapted.

Sample Path.

L^p integrable.

Def. 13.3. $\mathcal{F}_t$-adapted integrable process $X(t)$ is martingale w.r.t filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if.

$$E[X(t) \mid \mathcal{F}_s] = X(s)$$

Conditional expectation of next future value is equal to present value.

Doob's martingale inequality.

Let. $X(t)$: martingale, $p > 1$, $X(t) \in L^p(\Omega, \mathbb{R}^n)$

$$E\left[\sup_{a \leq t \leq b} \|X(t)\|^p\right] \leq \left(\frac{p}{p-1}\right)^p E\left[\|X(b)\|^p\right]$$

Borel - Cantelli's lemma.

If. $\{A_k\}_{k=1}^{\infty} \subset \mathcal{F}$ and. $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, then.

$$\mathbb{P}\left(\limsup_{k \to \infty} A_k\right) = 0$$

13.1. Existence & Uniqueness of Solution.

Thm 13.1 Assume.

(A1) f, g grow linearly :

$$\| f(t,x,y) \|^2 + \| g(t,x,y) \|^2 \le L_1 \left(1 + \|x\|^2 + \|y\|^2 \right)$$

(A2) f, g : global Lipschitz :

$$\| f(t,x_1,y_1) - f(t,x_2,y_2) \|^2 + \| g(t,x_1,y_1) - g(t,x_2,y_2) \|^2$$
$$\le L_2 \left(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 \right),$$

Then (SEPCA) has unique solution for $t \ge t_0$.

Proof).

(step 1) $\max_{1 \le n \le j} E[\|x_n(t)\|^2] \le C_2 e^{3L_1(t_1 - t_0 + 1)(t_1 - t_0)}$ for $t \in [t_0, t_1]$,

(We will define Cauchy sequence.)

(Step 2) $E[\|x_{n+1}(t) - x_n(t)\|^2] \le \dfrac{C[M(t-t_0)]^n}{n!}$, ($M$: to be determined.)

(step 3), $\lim_{n \to \infty} x_n(t) = x(t)$

Note. $a_n = \sqrt{n} \Rightarrow a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \to 0,$

but $\lim_{n \to \infty} a_n \to \infty.$

For $t \in [t_0, t_1)$, we define.

$$x_0(t) = x^0$$

$$\begin{cases} x_n(t) = x^0 + \displaystyle\int_{t_0}^{t} f(s, x_{n-1}(s), \lambda_0(x_{n-1}(t_0)))\, ds \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} g(s, x_{n-1}(s), \lambda_0(x_{n-1}(t_0)))\, dW(s) \\ x_n(0) = x^0. \end{cases}$$

$$E[\|x_n(t)\|^2] \leq 3E[\|x^0\|^2] + 3E\left[\left|\int_{t_0}^{t} f(s, \cdot, \cdot)\, ds\right|^2\right] + 3E\left[\left|\int_{t_0}^{t} g(s, \cdot, \cdot)\, dW(s)\right|^2\right]$$

$$\leq 3E[\|x^0\|^2]$$

$$+ 3(t_1 - t_0)\, E\left[\int_{t_0}^{t} |f(s, \cdot, \cdot)|^2 ds\right] \quad \text{Holder in } t.$$

$$+ 3\int_{t_0}^{t} E[|g(s, \cdot, \cdot)|^2]\, dt. \quad \text{Itô isometry}$$

(성장)

$$\leq 3E[\|x^0\|^2] + 3(t_1 - t_0)\int_{t_0}^{t_1} L_1 \left(1 + E[\|x_{n-1}(t)\|^2] \; E[\|\lambda_0(x_{n-1}(t_0))\|^2]\right)$$

$$+ 3L_1 \int_{t_0}^{t} \left(1 + E[\|x_{n-1}(t)\|^2] + E[\|\lambda_0(x_{n-1}(t_0))\|^2]\right)$$

$$\leq C_1 + 3L_1(t_1 - t_0 + 1)\int_{t_0}^{t_1} E[\|x_{n-1}(s)\|^2]\, ds.$$

$$\Rightarrow \max_{1\leq n\leq j} E[\|x_n(t)\|^2] \leq C_2 + 3L_1(t_1 - t_0 + 1)\max_{1\leq n\leq j}\int_{t_0}^{t_1} E[\|x_n(t)\|^2]\, ds.$$

By the Gronwall inequality

$$\max_{1\leq n\leq j} E[\|x_n(t)\|^2] \leq C_2\, e^{3L_1(t_1 - t_0 + 1)(t_1 - t_0)}$$

(Step 2) we use the induction argument.

when $n = 0$,

$$E[\|x_1(t) - x_0(t)\|^2] \leq 2E\left[\left|\int_{t_0}^{t} f(s, x_0(s), \lambda_0(x_0(t_0)))\, ds\right|^2\right]$$

$$+ 2E\left[\left|\int_{t_0}^{t} g(s, x_0(s), \lambda_0(x_0(t_0)))\, dw(s)\right|^2\right]$$

$$\leq 2L_1(t_1 - t_0)(1 + t_1 - t_0)$$

$$\times \left(1 + E[\|x^0\|^2] + E[\|\lambda_0(x^0)\|^2]\right)$$

$$= C.$$

Assume $n \leq k$ holds.

when $n = k+1$,

$$\underbrace{E\left[\|x_{n+2}(t)-x_{n+1}(t)\|^2\right]}_{A_{n+2}} \le \underbrace{2L_2(t-t_0+1)}_{:=M.}$$

$$\times E\left[\int_{t_0}^{t}\left(\underbrace{\|x_{n+1}(s)-x_n(s)\|^2}_{A_{n+1}} + \underbrace{\|\lambda_0(x_{n+1}(t_0))-\lambda_0(x_n(t_0))\|^2}_{=0}\right)ds\right]$$

$$A_{n+2}(t) \le M\int_{t_0}^{t} A_{n+1}(s)\,ds$$

$$= M\int_{t_0}^{t}\frac{C[M(s-t_0)]^n}{n!}\,ds$$

$$= \frac{C[M(t-t_0)]^{n+1}}{(n+1)!}.$$

(Step 3) (Step 2 implies)

$$E\left(\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\|^2\right) \le \frac{C\,M(t_1-t_0)^n}{n!}$$

$$\mathbb{P}\left\{\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\| > \frac{1}{2^n}\right\} \le \frac{C\,M(t-t_0)^n}{n!}.$$

By Borel-Cantelli's lemma,

$$\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\| \le \frac{1}{2^n}$$

$$x_n(t) = x_0(t) + \sum_{j=0}^{n-1} \underbrace{\left(x_{j+1}(t) - x_j(t)\right)}_{\text{geometric sequence.}}$$

there exists limit point.

$$\Rightarrow \quad \lim_{n\to\infty} \| x_n(t) - x(t) \|_{L^2} \to 0$$

$$x(t) \overset{?}{=} x^0 + \int_{t_0}^{t} f(s, x(s), \lambda_0(x(t_0)))\, ds$$

$$\Big\downarrow \text{ in } L^2 \qquad + \int_{t_0}^{t} q(s, x(s), \lambda_0(x(t_0)))\, dW(s)$$

$$x_{n+1}(t) = x^0 + \int_{t_0}^{t} f(s, x_n(s), \lambda_0(x_n(t_0)))\, ds$$

$$+ \int_{t_0}^{t} q(s, x_n(s), \lambda_0(x_n(t_0)))\, ds,$$

$$\text{RHS} \le L_2(t_1 - t_0 + 1) \int_{t_0}^{t_1} E\left[\| x(s) - x_n(s) \|^2 \right] ds$$

13.2 Comparison method.

Thm 13.2 Assume

(i). for any $K = 0, 1, 2, \cdots$, $V \in C^{1,2}([t_k, t_{k+1}) \times \mathbb{R}^n ; \mathbb{R})$

V is bounded below &

$$\mathcal{L} V(t, x, y) \le h(t, x, \sigma_k(y)) \quad a.s \quad t \in [t_k, t_{k+1})$$

where h: concave, non decreasing in x, σ_k,

σ_k: concave function

(ii) $\dot{u}(t) = h(t, u(t), \sigma_k(u(t_k)))$ $\quad t \in [t_k, t_{k+1})$

$u(t_0) = u_0$

has maximal solution $r(t; t_0, u_0)$ for all $t \ge t_0$.

$\Rightarrow$ $E[V(t_0, x_0)] \le u_0$ implies $E[V(t, x)] \le r(t; t_0, x_0)$

13.3 Stability Analysis.

Thm 13.3. Suppose the same assumption of Thm 13.2.

Let $\exists a, b \in K$ s.t

$$b(\|x\|^2) \leq V(t,x) \leq a(\|x\|^2),$$

Then stability property of $u=0$ imply

stability of $x=0$ of (SEPCA).